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On The Binary Quadratic Diophantine Equation $x^2 - 9xy + y^2 + 21x = 0$

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Abstract -The binary quadratic Diophantine equation $x^2 - 9xy + y^2 + 21x = 0$ is studied for its non-trivial integral solutions. The recurrence relations satisfied by the solutions x and y are given. A few interesting properties among the solutions are presented.

Index Terms - Binary quadratic equation, Integral solutions, Pell equation, Diophantine equation.

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1. INTRODUCTION

The binary quadratic diophantine equations (both homogeneous and non homogeneous) are rich in variety [1-6]. In [7-18] the binary quadratic non-homogeneous equations representing hyperbolas are studied for their non–zero integral solutions. These results have motivated us to search for infinitely many non-zero integral solutions of the binary quadratic equation given by $x^2 - 9xy + y^2 + 21x = 0$. The recurrence relations satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited.

2. METHOD OF ANALYSIS

The diophantine equation representing the binary quadratic equation to be solved for its non-zero distinct integral solutions is

$$x^2 - 9xy + y^2 + 21x = 0 \tag{1}$$

Note that (1) is satisfied by the following non-zero distinct integer pair

(3, 3), (-21,-189), (-1701,-189)

However, we have two more patterns of solutions for (1), which are illustrated below:

Pattern: I

Solving (1) for y:

$$y = \frac{9x \pm \sqrt{81x^2 - 4(x^2 + 21x)}}{2} \tag{2}$$

Let $x = 2X$ (3)

Then $y = 9x \pm \sqrt{77X^2 - 42X}$ (4)

Assuming $77X^2 - 42X = \beta^2$ (5)

(4) becomes $y = 9X \pm \beta$ (6)

(5) is written as

$$S^2 = 77\beta^2 + 21^2 \tag{7}$$

where $S = 77X - 21$ (8)

Now, consider the Pellian equation

$$S^2 = 77\beta^2 + 1 \tag{9}$$

whose general solution $(\tilde{S}_n, \tilde{\beta}_n)$ is given

$$\tilde{S}_n = \frac{1}{2}\left[(351 + 40\sqrt{77})^{n+1} + (351 - 40\sqrt{77})^{n+1}\right]$$

$$\tilde{\beta}_n = \frac{1}{2\sqrt{77}}\left[(351 + 40\sqrt{77})^{n+1} - (351 - 40\sqrt{77})^{n+1}\right], n = 0,1,2.........$$

Thus, the general solution (S_n, β_n) of (7) is obtained as

$$S_n = \frac{21}{2}\left[(351 + 40\sqrt{77})^{n+1} + (351 - 40\sqrt{77})^{n+1}\right] \tag{10}$$

$$\beta_n = \frac{21}{2\sqrt{77}}\left[(351 + 40\sqrt{77})^{n+1} - (351 - 40\sqrt{77})^{n+1}\right], n = 0,1,2..... \tag{11}$$

In view of (3), (8) and (10), we have

$$x_n = \frac{3}{11}\left(\frac{f_n}{2} + 1\right) \tag{12}$$

where

$$f_n = (351+40\sqrt{77})^{n+1} + (351-40\sqrt{77})^{n+1} \quad (13)$$

Again, in view of (3), (6) and (11), we have

$$y_n = \frac{189}{77}\left(\frac{f_n}{2}+1\right) \pm 21\left(\frac{g_n}{2\sqrt{77}}\right) \quad (14)$$

where

$$g_n = (351+40\sqrt{77})^{n+1} - (351-40\sqrt{77})^{n+1} \quad (15)$$

Our aim is to get integer solutions to (1), which is obtained for $n = 0, 2, 4, ...$

Hence, we have

$$x_{2n} = \frac{3}{11}\left(\frac{F_{2n}}{2}+1\right) \quad (16)$$

$$y_{2n} = \frac{189}{77}\left(\frac{F_{2n}}{2}+1\right) \pm 21\left(\frac{G_{2n}}{2\sqrt{77}}\right) \quad (17)$$

where

$$F_{2n} = (351+40\sqrt{77})^{2n+1} + (351-40\sqrt{77})^{2n+1} \quad (18)$$

$$G_{2n} = (351+40\sqrt{77})^{2n+1} - (351-40\sqrt{77})^{2n+1} \quad (19)$$

Equations (16) and (17) together will give the distinct integral solutions of (1).

The above values of x_{2n} and y_{2n} satisfy respectively the following recurrence relations

$$x_{2n+4} - 492802x_{2n+2} + x_{2n} = -268800 \quad (20)$$

$$y_{2n+4} - 492802y_{2n+2} + y_{2n} = -1209600 \quad (21)$$

A few numerical examples are given below

n	x_{2n}	y_{2n}
0	192	1704, 24
1	94348992	838524984, 10615944
2	46495371686592	413226787953864, 5231557225464

A few interesting relations among the solutions are presented below.

1. The values of x, y both are positive and even.

2. $x_{2n} - y_{2n} \equiv 0 \pmod{24}$

3. $x_{2n} + y_{2n} \equiv 0 \pmod{27}$

4. $x_{2n+2} - 56160y_{2n} + 6319x_{2n} = -134400$

5. $y_{2n+2} - 499121y_{2n} + 56160x_{2n} = -1194480$

6. $492800x_{2n} + 492801x_{2n+2} - x_{2n+4} = 537600$

7. $492800y_{2n} + 492801y_{2n+2} - y_{2n+4} = 2420880$

8. The following expressions are nasty numbers

 i) $126[492800x_{2n} + 492801x_{2n+2} - x_{2n+4}]$

 ii) $18[x_{2n+4} - y_{2n+4} + 492802(y_{2n+2} - x_{2n+2}) + x_{2n} - y_{2n}]$

9. If (x_0, y_0) be any given solution to (1), then

 $(3-y_0, 3-x_0)$ is also a solution to (1).

3. REMARKABLE OBSERVATIONS

1) By considering suitable linear transformations between the solutions of (1), one may get integer solutions for the hyperbola.

Example 1:

Define $X = 11x_{2n} - 6$, $Y = 22y_{2n} - 99x_{2n}$

Note that the pair (X, Y) satisfies the hyperbola

$$Y^2 = 77X^2 - 2772$$

By considering suitable linear transformations between the solutions of (1), one may get integer solutions for parabola.

Example 2:

Define $X = 11x_{2n} - 6$, $Y = 22y_{2n} - 99x_{2n}$

Note that the pair (X, Y) satisfies the parabola

$$Y^2 = 231X - 2772$$

Pattern II:

Solving (1) for x

$$x = \frac{1}{2}\left[(9y-21) \pm \sqrt{77y^2 - 378y + 441})\right] \quad (22)$$

Let $\beta^2 = 77y^2 - 378y + 441$

Multiplying the above equation by 77 on both sides and performing a few calculations, we have

$$S^2 = 77\beta^2 + 1764 \tag{23}$$

where $S = 77y - 189$ (24)

The positive integer solution of (23) is

$$\beta_0 = 21, S_0 = 189$$

Now to find the other solution of (23), consider the Pellian equation

$$S^2 = 77\beta^2 + 1 \tag{25}$$

Whose fundamental solution is

$$(\tilde{\beta}_0, \tilde{S}_0) = (40,351)$$

The other solutions of (25) can be derived from the relations

$$\tilde{S}_n = \frac{f_n}{2}\ ,\quad \tilde{\beta}_n = \frac{g_n}{2\sqrt{77}}$$

Where

$$f_n = (351 + 40\sqrt{77})^{n+1} + (351 - 40\sqrt{77})^{n+1}$$

$$g_n = (351 + 40\sqrt{77})^{n+1} - (351 - 40\sqrt{77})^{n+1}$$

Applying the lemma of Brahmagupta between (β_0, S_0) & $(\tilde{\beta}_n, \tilde{S}_n)$, the other solutions of (23) can be obtained from the relations

$$\beta_{n+1} = \frac{21}{2} f_n + \frac{189}{2\sqrt{77}} g_n \tag{26}$$

$$S_{n+1} = \frac{1}{2}(189 f_n + 21\sqrt{77} g_n) \tag{27}$$

Taking positive sign on the R.H.S of (22) and using (24), (26) & (27), the non-zero distinct integer solutions of the hyperbola (1) are obtained as follows

$$x_{n+1} = \frac{1}{2}\left(9y_{n+1} - 21 + \frac{21}{2} f_n + \frac{189}{2\sqrt{77}} g_n\right) \tag{28}$$

$$y_{n+1} = \frac{3}{11}\left(9 f_n + \sqrt{77} g_n + 9\right),\ n=-2, 0, 2 \ldots \tag{29}$$

The recurrence relations satisfied by x_{n+1}, y_{n+1} are respectively

$$x_{n+5} - 492802x_{n+3} + x_{n+1} = -268800$$

$$y_{n+5} - 492802y_{n+3} + y_{n+1} = -1209600$$

A few numerical examples are presented in the table below

n	x_{n+1}	y_{n+1}
-2	3, 192	24
0	15123, 192	1704
2	7452375843, 94348992	838524984
4	367254571989816, 46495371686592	413226787953864

A few interesting relations among the solutions are presented below.

1) The values of x, y both are positive.

2) $x_{n+1} + y_{n+1} \equiv 0 \pmod 3$

3) $x_{n+3} - 499121x_{n+1} + 56160y_{n+1} = -134400$

4) $y_{n+3} - 56160x_{n+1} + 6319y_{n+1} = -15120$

5) $56160x_{n+3} - 499121y_{n+3} + y_{n+1} = -1194480$

6) $6319x_{n+3} - 56160y_{n+3} + x_{n+1} = -134400$

7) $\frac{1}{6}\left[198y_{3n+3} - 22x_{3n+3} - 474\right] + 3\left[\frac{1}{6}(198y_{n+1} - 22x_{n+1} - 474)\right]$

is a cubic integer.

8) $18\left[x_{n+5} - y_{n+5} + 492802(y_{n+3} - x_{n+3}) + x_{n+1} - y_{n+1}\right]$

is a nasty number.

4. REMARKABLE OBSERVATIONS

1) By considering suitable linear transformations between the solutions of (1), one may get integer solutions for the hyperbola.

Example 3:

Define $X = 198y_{n+1} - 22x_{n+1} - 474$,

$$Y = 198x_{n+1} - 1738y_{n+1} + 4158$$

Note that the pair (X, Y) satisfies the hyperbola

$$Y^2 = 77X^2 - 11088$$

By considering suitable linear transformations between the solutions of (1), one may get integer solutions for parabola.

Example 4:

Define $X = 198y_{n+1} - 22x_{n+1} - 474$,

$$Y = 198x_{n+1} - 1738y_{n+1} + 4158$$

Note that the pair (X, Y) satisfies the parabola

$$Y^2 = 6X - 11088$$

5. CONCLUSION

In this paper, we have made an attempt to obtain a complete set of non-trivial distinct solutions for the Non-homogeneous binary quadratic equation. To conclude, one may search for other choices of solutions to the considered binary equation and further, quadratic equations with multi-variables.

REFERENCES

[1] Banumathy,T.S,(1995) A Modern Introduction To Ancient Indian Mathematics,Wiley Eastern Limited, London.

[2] Carmichael,R.D,(1950) The Theory Of Numbers And Diophantine Analysis,Dover Publications,New York.

[3] Dickson.L.E,(1952)History Of The Theory Of Numbers,Vol.II,Chelsia Publicating Co,New York.

[4] Mordell,L.J,(1969)Diophantine Equations Acadamic Press,London.

[5] Nigel,P.Smar.,(1999)The Algorithm Resolutions Of Diophantine Equations, Cambridge University,Press,London.

[6] Telang,S.G,(1996)Number Theory,Tata,Mc Graw-Hill Publishing Company,New Delhi.

[7] Gopalan.M.A.,and Parvathy.G,(2010) "Integral Points On The Hyperbola $x^2 + 4xy + y^2 - 2x - 10y + 24 = 0$'Antarctica J.Math,Vol 1(2),149-155.

[8] Gopalan.M.A.,Vidhyalakshmi.S.,Sumathi.G and Lakshmi.K sep (2010)"Integral Points On The Hyperbola $x^2 + 6xy + y^2 + 40x + 8y + 40 = 0$",BesselJ.Math. Vol 2(3),159-164.

[9] Gopalan.M.A.,Vidhyalakshmi.S,(2007)"On The Diophantine Equation $x^2 + 4xy + y^2 - 2x + 2y - 6 = 0$, ActacineciaIndia, Vol.xxxIIIM.NO2, P.567-570.

[10] Gopalan.M.A.,Gokila.,and Vidhyalakshmi.S.,and Devibala.S,(2007) "On The Diophantine Equation $3x^2 + xy = 14$",Acta Ciencia India, Vol.xxxIIIM.NO,P.645-646.

[11] Gopalan.M.A.,and Janaki.G,(2008)"Observation On $x^2 - y^2 + x + y + xy = 2$ ", Impact J.sci.,Tech,Vol 2(3) P.143-148.

[12] Gopalan.M.A., Shanmuganadham.P., and Vijayashankar.A,(2008)"On Binary Quadratic Equation" $x^2 - 5xy + y^2 + 8x - 20y + 15 = 0$", Acta cinecia India, Vol.xxxIVM.NO.4,P.1803-1805.

[13] Gopalan.M.A.,Vidhyalakshmi.S., Lakshmi.K, and Sumathi.G,sep(2012)" Observation On $3x^2 + 10xy + 4y^2 - 4x + 2y - 7 = 0$ Diophantus J.Maths.Vol.1 (2), 123-125.

[14] Mollion.R.A,(1998)"All Solutions Of The Diophantine Equations $x^2 - Dy^2 = n$" Far East J,Math.Sci.,Speical Volume,Part III,P.257,293.

[15] Vidhyalakshmi.S., Gopalan.M.A.,and Lakshmi.K, (2014)" On Binary Quadratic Equation" $3x^2 - 8xy + 3y^2 + 2x + 2y + 6 = 0$ Scholar Journal Of Physics,Mathematics And Statistics, Vol.1(2),(sep-nov),41-45.

[16] Vidhyalakshmi.S., Gopalan.M.A.,and Lakshmi.K August (2014)"Integer Solutions of the Binary Quadratic Equation $x^2 - 5xy + y^2 + 33x = 0$", International Journal of Innovative Science Engineering&Technology, Vol.1 (6), 450-453.

[17] Vidhyalakshmi.S., Gopalan.M.A., and Nandhini.S, July (2015) "On the Binary Quadratic Equation $x^2 - 3xy + y^2 + 18x = 0$"International Research Journal of EngineeringandTechnology, Vol.2 (4), 825- 829.

[18] Gopalan.M.A.,Geetha.V.,and Priyanka.D, June(2015) "On The Binary Quadratic Diophantine Equation $x^2 - 3xy + y^2 + 16x = 0$"International Journal Of Scientific Engineering and Applied Science, Vol.1 (4), 516-520.